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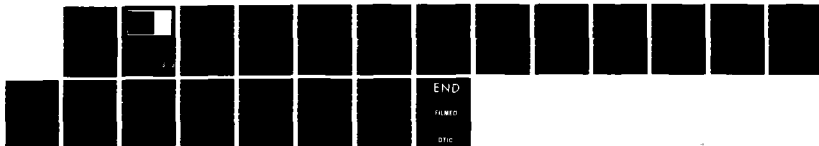
ABOUT PRODUCT FORMULAS WITH VARIABLE STEP-SIZE(U)
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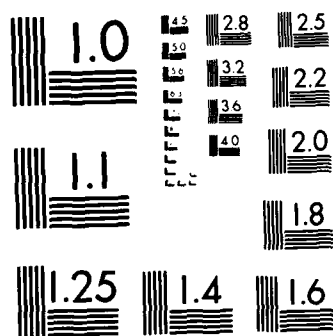
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ABOUT PRODUCT FORMULAS WITH
VARIABLE STEP-SIZE

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UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

ABOUT PRODUCT FORMULAS WITH VARIABLE STEP-SIZE

Michel Pierre* and Mounir Rihani*

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ABSTRACT

A continuous semigroup of linear contractions $(S(t))_{t \geq 0}$ on a Banach space X can be generated through the product formula

$$(1) \quad S(t)x = \lim_{n \rightarrow \infty} U\left(\frac{t}{n}\right)^n x \quad \forall x \in X \quad \forall t \geq 0$$

where $(U(h))_{h \geq 0}$ is any family of contractions on X whose right-derivative at $h = 0$ coincides with the infinitesimal generator A of $S(t)$, that is

$$(2) \quad \forall x \in D(A) \quad \lim_{h \rightarrow 0} \frac{x - U(h)x}{h} = Ax \quad (= \lim_{t \rightarrow 0} \frac{x - S(t)x}{t}).$$

Actually formula (1) extends to the case when $S(t)$ is a semigroup of nonlinear contractions "generated" by an m -accretive operator A and condition (2) may also be weakened to

$$(3) \quad \frac{I - U(h)}{h} \xrightarrow{h \rightarrow 0} A \quad \text{in the sense of graphs.}$$

We look here at the same formula (1) when the regular step-size t/n is replaced by a variable step-size, namely

$$(4) \quad \lim_{n \rightarrow \infty} U\left(\frac{h_n}{N}\right) U\left(\frac{h_n}{N-1}\right) \dots U\left(\frac{h_1}{1}\right)x = S(t)x \quad \forall x \in \overline{D(A)}$$

where $\sum_{1 \leq i \leq N^n} h_i^n = t$ and $\lim_{n \rightarrow \infty} \left(\max_{1 \leq i \leq N^n} h_i^n \right) = 0$.

Surprisingly, it turns out that (4) fails under assumption (3); we exhibit a counterexample. However, we prove that (4) holds true under the stronger assumption (2).

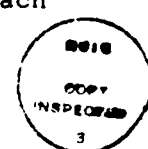
AMS (MOS) Subject Classification: 47H20, 65J15, 35R15.

Key Words: Product formulas, nonlinear semigroups, accretive operators, approximation scheme, stability and convergence.

Work Unit Number 1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Many numerical schemes are based on the following abstract product formula: given an evolution equation which can be written in the abstract form

$$\frac{du}{dt}(t) + Au(t) = 0 \quad \text{on } (0, T)$$

$$u(0) = u_0 \quad \text{given}$$

with A being a suitable operator on a functional space, then under adequate assumptions the solution u is given by the formula

$$u(t) = \lim_{n \rightarrow \infty} U\left(\frac{t}{n}\right)^n u_0$$

where $(U(h))_{h > 0}$ is a family of mappings whose right-derivative at $h = 0$ coincides with A , that is

$$\forall x \in D(A) \quad \lim_{h \rightarrow 0} \frac{x - U(h)x}{h} = Ax.$$

This is a natural extension of the exponential formula

$$u(t) = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n} A\right)^{-n} u_0.$$

Above product formula provides an extensively used tool for the Lie-Trotter formulas insuring the convergence of various alternating direction methods for evolution equations.

In view of numerical applications, it is important to relax the restriction that the time step-size be regular in the above schemes when, for instance, coupled with path control technics or even for stability requirements.

The goal of this paper is to decide whether the Chernoff formula is valid when the step-size is variable. It turns out that under the usual assumptions instability and lack of convergence can appear as soon as the step size is slightly perturbed. We give here an example in that direction. However, we prove that slightly stronger assumptions can prevent these pathological situations.

Michel Pierre* and Mounir Rihani*

1. Introduction.

Given $(S(t))_{t \geq 0}$ a continuous semi-group of linear contractions in a Banach space X , it can be generated via the following exponential formula:

$$(1) \quad \forall t \geq 0; \forall x \in X, \quad S(t)x = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n} A\right)^{-n} x$$

where $-A$ denotes its infinitesimal generator and I is the identity. More generally, one can replace in (1) the resolvents $(I + h A)^{-1}$ by a family of contractions $(U(h))_{h \geq 0}$ from X into X satisfying

$$(2) \quad \forall x \in D(A) \quad \lim_{h \rightarrow 0} \frac{x - U(h)x}{h} = Ax.$$

Thus, it has been proved (see [11], [3]) that (2) implies

$$(3) \quad \forall t \geq 0, \forall x \in X, \quad \lim_{n \rightarrow \infty} U\left(\frac{t}{n}\right)^n x = S(t)x.$$

Actually, as proved in [7], [4], the condition (2) can be weakened to assuming that

$$(4) \quad \forall x \in X, \forall \lambda > 0, \quad \lim_{h \rightarrow 0} \left(I + \lambda \frac{I - U(h)}{h}\right)^{-1} x = (I + \lambda A)^{-1} x$$

which is equivalent to saying that the graph in $X \times X$ of the operator $h^{-1}(I - U(h))$ converges to the graph of A as h tend to 0.

The implication (4) \Rightarrow (3) turns out to have a nonlinear version as proved by Brézis - Pazy [2]. Namely, let A be an m-accretive operator in X , that is $A \subset X \times X$ and

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(5) $\forall \lambda > 0, (I + \lambda A)^{-1}$ is a contraction everywhere defined on X .

Let $(S(t))_{t \geq 0}$ be the nonlinear semigroup generated by A in the sense of Crandall - Liggett [5], that is

$$(6) \quad \forall t \geq 0, \forall x \in \overline{D(A)}, S(t)x = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n} A \right)^{-n} x.$$

Then, if $(U(h))_{h \geq 0}$ is a family of contractions from X into X satisfying (4), we have the product formula

$$(7) \quad \forall t \geq 0, \forall x \in \overline{D(A)}, \lim_{n \rightarrow \infty} U\left(\frac{t}{n}\right)^n x = S(t)x.$$

The purpose of this paper is to look at the formula (7) when the regular step size $\frac{t}{n}$ is replaced by a variable step-size. In particular, under the same assumptions as above does one have

$$(8) \quad \lim_{n \rightarrow \infty} U(h_{N_n}^n) U(h_{N_n-1}^n) \dots U(h_1^n) x = S(T)x$$

when

$$\sum_{i=1}^{N_n} h_i^n = T, \quad \lim_{n \rightarrow \infty} \left(\max_{1 \leq i \leq N_n} h_i^n \right) = 0?$$

We give here a negative and a positive answer to that question:

a) given A an m -accretive operator the condition (4) does not imply (8) as proved by a counterexample described in paragraph 3.

b) if A is a single-valued m -accretive operator and if $U(h)$ satisfies the stronger assumption (2), then (8) holds.

Acknowledgements. The authors thank Mike Crandall for several helpful discussions on that subject.

2. The positive result.

Let X be a Banach space with its norm denoted by $|\cdot|$ and let A be a single-valued, accretive operator on X , that is

$$(9) \quad \forall \lambda > 0, \forall x, \hat{x} \in D(A), |x - \hat{x}| \leq |x - \hat{x} + \lambda(Ax - A\hat{x})|.$$

For $h \in (0, d)$ let $U(h): \overline{D(A)} \rightarrow \overline{D(A)}$ satisfy

$$(10) \quad \forall h \in (0, h_0), \forall x, \hat{x} \in \overline{D(A)} \quad |U(h)x - U(h)\hat{x}| \leq |x - \hat{x}|$$

$$(11) \quad \forall x \in D(A), \quad \lim_{h \rightarrow 0} \frac{x - U(h)x}{h} = Ax.$$

Given $T > 0$, σ a subdivision of $(0, T)$:

$$(12) \quad 0 = t_0 < t_1 < \dots < t_{N-1} < T \leq t_N, \quad |\sigma| = \max_{1 \leq i \leq N} (t_i - t_{i-1})$$

and $z: [0, T[\rightarrow X$ such that

$$(13) \quad \forall i = 1, \dots, N, \forall t \in]t_{i-1}, t_i], z(t) = z_i \in X,$$

let us assume there exist x_0, x_1, \dots, x_N in $\overline{D(A)}$ such that

$$(14) \quad \forall i = 1, \dots, N, \quad x_i - U(h_i)x_{i-1} = h_i z_i$$

where $h_i = t_i - t_{i-1}$. We then define v_{σ, z, x_0} by

$$(15) \quad v_{\sigma, z, x_0}(0) = x_0, \quad \forall i = 1, \dots, N, \forall t \in]t_{i-1}, t_i], v_{\sigma, z, x_0}(t) = x_i.$$

Remark. If $z \equiv 0$, the relation (14) always defines a sequence (x_i) such that $x_i = U(h_i)U(h_{i-1})\dots U(h_1)x_0$ which are the expressions we are mainly interested in. However, many situations involve perturbed schemes like (14) where z is not zero but of small norm in $L^1(0, T; X)$ and our method applies as well to this more general situation.

Our goal is to describe the behavior of v_{σ^n, z^n, x^n} when

$$\lim_{n \rightarrow \infty} |\sigma^n| = \lim_{n \rightarrow \infty} \int_0^T |z^n(t)| dt = \lim_{n \rightarrow \infty} |x^n - x_0| = 0.$$

We expect it to tend to the solution (if it exists) of

$$(16) \quad \frac{du}{dt} + Au = 0 \quad \text{on } (0, T), \quad u(0) = x_0 \in \overline{D(A)}.$$

By solution of (16), we mean a function $u \in C([0, T]; X)$ such that there exists a sequence of step-functions $u^p [0, T] \rightarrow X$ satisfying

$$(17) \quad \forall j = 1, \dots, N^p, \quad \forall s \in [s_{j-1}^p, s_j^p], \quad u^p(s) = u_j^p$$

where

$$(18) \quad 0 = s_0^p < s_1^p < \dots < s_{N^p-1}^p < T < s_{N^p}^p, \quad \ell_j^p = s_j^p - s_{j-1}^p$$

$$(19) \quad \frac{u_j^p - u_{j-1}^p}{\ell_j^p} + Au_j^p = w_j^p \quad (u_0^p = u^p(0))$$

$$(20) \quad 0 = \lim_{p \rightarrow \infty} \sup_{0 \leq s \leq T} |u^p(s) - u(s)| = \lim_{p \rightarrow \infty} \sum_{j=1}^{N^p} \ell_j^p |w_j^p| = \lim_{p \rightarrow \infty} \max_{1 \leq j \leq N^p} \ell_j^p.$$

We will refer to such a solution as a mild solution of (16) (see [1]).

Theorem 1. Let u be a mild solution of (16). Let $v^n = v_{\sigma^n, z^n, x^n}$ be a sequence of functions defined by (12) - (15) where

$$\lim_{n \rightarrow \infty} |\sigma^n| = \lim_{n \rightarrow \infty} \int_0^T |z^n(t)| dt = \lim_{n \rightarrow \infty} |x^n - x_0| = 0.$$

Then, if (9), (10), (11) hold

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} |u(t) - v^n(t)| = 0.$$

In the case when the range of $I + \lambda A$ contains $\overline{D(A)}$ for all $\lambda > 0$, thanks to Crandall - Liggett [5], one knows that

$$(21) \quad \forall x \in \overline{D(A)}, \quad \forall t > 0, \quad S(t)x = \lim_{n \rightarrow \infty} \left(I + \frac{t}{n}A\right)^{-n}x \text{ exists,}$$

and $S(t)$ is a semigroup of nonlinear contractions from $\overline{D(A)}$ into itself. Moreover, by definition, $t \mapsto S(t)x_0$ is a mild solution of (16) for all $T > 0$.

In that situation, as an immediate corollary of Theorem 1, we have

Proposition 1. Assume A satisfies (9), (21) and that (10), (11) hold. Then if $(h_i^n)_{1 \leq i \leq N^n}$ is a sequence such that

$$(22) \quad \lim_{n \rightarrow \infty} \left(\max_{1 \leq i \leq N^n} h_i^n\right) = 0, \quad \lim_{n \rightarrow \infty} \sum_{i=1}^{N^n} h_i^n = t$$

we have for all $x_0 \in \overline{D(A)}$

$$(23) \quad \lim_{n \rightarrow \infty} U(h_{N^n}^n) U(h_{N^{n-1}}^n) \dots U(h_1^n)x_0 = S(t)x_0$$

and the convergence is uniform in t on bounded intervals.

The proof of Theorem 1 is based on the next lemma which is an adaptation of technics used in [6] and [8].

Lemma 1. Let $u^P: [0, T] \rightarrow X$ satisfying (17) - (19). Then, for all $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, p) > 0$ such that for all functions v_{σ, z, x_0} satisfying (12) - (15) with $|\sigma| < \eta$ we have

$$\forall y \in D(A), \quad \forall i = 0, \dots, N, \quad \forall j = 0, \dots, N^P$$

$$(24) \quad |x_i - u_j^P| \leq |x_0 - y| + |u_0^P - y| + \{(t_i - s_j^P)^2 + |\sigma|t_i + |\sigma^P|s_j^P\}^{1/2} M_c$$

$$+ \sum_{k=1}^i h_k(|z_k| + \varepsilon) + \sum_{k=1}^j \ell_k^P |w_k^P|,$$

where $|\sigma^p| = \max_{1 \leq j \leq N^p} \ell_j^p$ and

$$(25) \quad M_\sigma(y) = \max \{ |Ay|, \max_{1 \leq i \leq N} h_i^{-1} |y - U(h_i)y| \}.$$

Proof. Let us denote $A^h = h^{-1}(I - U(h))$. By (11), for $\varepsilon > 0$ and p fixed, there exists $\eta = \eta(\varepsilon, p) > 0$ such that

$$(26) \quad h \in]0, \eta] \Rightarrow \forall j = 1, \dots, N^p, |A^h u_j^p - Au_j^p| < \varepsilon.$$

Now, we will assume $|\sigma| < \eta$ and, since p is fixed we will drop the indexation by p .

Let us denote $a_{i,j} = |x_i - u_j|$ and let us establish (24) in four steps. We fix $y \in D(A)$.

i) estimate of $a_{0,j}$: since A is accretive and by (19)

$$|u_j - y| \leq |u_j - y + \ell_j(Au_j - Ay)| \leq |u_{j-1} - y| + \ell_j w_j + \ell_j |Ay|.$$

By induction, for all $j \geq 1$

$$|u_j - y| \leq |u_0 - y| + \sum_{k=1}^j \ell_k w_k + s_j |Ay|.$$

Hence, for all $j \geq 0$

$$(27) \quad a_{0,j} \leq |x_0 - y| + |u_0 - y| + s_j |Ay| + \sum_{k=1}^j \ell_k w_k$$

which implies (24) for $i = 0$ and $j = 0, \dots, N^p$.

ii) estimate of $a_{i,0}$:

$$\begin{aligned} |x_i - y| &\leq |x_i - U(h_i)x_{i-1}| + |U(h_i)x_{i-1} - U(h_i)y| + |U(h_i)y - y| \\ &\leq h_i |z_i| + |x_{i-1} - y| + h_i M_\sigma(y). \end{aligned}$$

By induction, for all $i \geq 1$

$$|x_i - y| \leq |x_0 - y| + \sum_{k=1}^i h_k |z_k| + t_i M_\sigma(y) .$$

Hence for all $i \geq 0$

$$(28) \quad a_{i,0} \leq |u_0 - y| + |x_0 - y| + \sum_{k=1}^i h_k |z_k| + t_i M_\sigma(y) .$$

iii) estimate of $a_{i,j}$ in terms of $a_{i-1,j}$ and $a_{i,j-1}$: for all $i, j \geq 1$

$$(29) \quad (1 + \ell_j h_i^{-1}) a_{i,j} = |x_i - u_j + \ell_j h_i^{-1} (x_i - U(h_i) u_j + U(h_i) u_j - u_j)| .$$

Using (14), we have

$$|x_i - U(h_i) u_j| \leq a_{i-1,j} + h_i |z_i| .$$

Plugging this inequality in (29) together with (26) applied with $h = h_i < \eta$ leads to

$$(30) \quad (1 + \ell_j h_i^{-1}) a_{i,j} \leq \ell_j h_i^{-1} a_{i-1,j} + \ell_j |z_i| + \ell_j \varepsilon + |x_i - u_j - \ell_j A u_j| .$$

Now by (19)

$$(31) \quad |x_i - u_j - \ell_j A u_j| \leq a_{i,j-1} + \ell_j |w_j| .$$

Finally, (30) and (31) yield

$$(33) \quad a_{i,j} \leq \frac{\ell_j}{h_i + \ell_j} a_{i-1,j} + \frac{h_i}{h_i + \ell_j} a_{i,j-1} + \frac{\ell_j h_i}{h_i + \ell_j} (|z_i| + |w_j| + \varepsilon) .$$

iv) induction procedure: we assume that (24) holds with (i,j) replaced by $(i-1,j)$ and (i,j) . Plugging it into (33) gives (24) after a easy computation. Thanks to the two first steps, we deduce that (24) holds for all (i,j) .

Proof of Theorem 1. Let $v^n = v_{\sigma^n, z^n, x^n}$ be a sequence of functions defined by (12) - (15) where

$$\lim_{n \rightarrow \infty} |\sigma^n| = \lim_{n \rightarrow \infty} \int_0^T |z^n(t)| dt = \lim_{n \rightarrow \infty} |x^n - x_0| = 0.$$

Let u be a mild solution of (16) and let u^p be an approximated sequence as in (17) - (20). Let us denote

$$|\hat{\sigma}^p| = \max_{1 \leq j \leq N^p} \ell_j^p.$$

By lemma 1, for all $p > 1$ and $\varepsilon > 0$, there exists $N(p)$ such that

$$\forall n > N(p) \quad \forall 1 \leq i \leq N^n \quad \forall 1 \leq j \leq N^p \quad \forall y \in D(A)$$

$$\begin{aligned} |x_i^n - u_j^p| &\leq |x^n - y| + |u_0^p - y| + \{ (t_i^n - s_j^p)^2 + |\sigma^n| t_i^n + |\hat{\sigma}^p| s_j^p \}^{1/2} M_\infty(y) \\ &\quad + \sum_{k=1}^i h_k^n (|z_k^n| + \varepsilon) + \sum_{k=1}^j \ell_k^p |w_k^p| \end{aligned}$$

where

$$M_\infty(y) = \sup_{h \in (0,1)} |h^{-1}(y - U(h)y)|.$$

Using

$$|v^n(t) - u(t)| \leq |v^n(t) - u^p(t)| + |u^p(t) - u(t)|$$

we deduce that for all $p > 0$ and $\varepsilon > 0$:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left(\sup_{0 \leq t \leq T} |v^n(t) - u(t)| \right) &\leq \sup_{0 \leq t \leq T} |u^p(t) - u(t)| + |x_0 - y| + |u_0^p - y| \\ &\quad + \{ |\hat{\sigma}^p|^2 + T |\hat{\sigma}^p| \}^{1/2} M_\infty(y) + \varepsilon T + \sum_{k=1}^{N^p} \ell_k^p |w_k^p|. \end{aligned}$$

Now letting p tend to ∞ , then ε tend to 0 yield (according to (20))

$$\limsup_{n \rightarrow \infty} \left(\sup_{0 \leq t \leq T} |v^n(t) - u(t)| \right) \leq 2|x_0 - y|.$$

Since $x_0 \in \overline{D(A)}$ and y is arbitrary in $D(A)$, the conclusion of the theorem follows.

3. The counterexample.

Let $X = C_0(\mathbb{R})$ the space of continuous functions u on \mathbb{R} such that $\lim_{x \rightarrow \pm\infty} u(x) = 0$, equipped with the norm

$$\|u\|_{\infty} = \sup_{x \in \mathbb{R}} |u(x)|.$$

we define

$$D(A) = \{u \in C_0(\mathbb{R}); u' \in C_0(\mathbb{R})\}$$

$$\forall u \in D(A), \quad Au = u'.$$

Then, A is the infinitesimal generator of the semigroup $S(t)$ on X defined by

$$\forall u \in C_0(\mathbb{R}), \quad S(t)u(x) = u(x - t)$$

which means that one has

$$\forall u \in D(A), \quad \lim_{t \rightarrow 0} \frac{u(x) - S(t)u(x)}{t} = u'(x) \quad \text{uniformly on } \mathbb{R}.$$

Now, let $u_0 \in X$ and $\alpha: (0,1) \rightarrow \mathbb{R}$ such that

$$(34) \quad \lim_{h \rightarrow 0} \alpha(h) = 0.$$

We set

$$(35) \quad \forall h \in (0,1), \quad \forall u \in X, \quad U(h)u = S(h)u + \alpha(h)(u_0 - S(h)u_0).$$

Then we have

Theorem 2.

1) Under assumption (34)

$$\forall h \in (0,1), \quad \forall u, \hat{u} \quad \|U(h)u - U(h)\hat{u}\|_{\infty} \leq \|u - \hat{u}\|_{\infty}$$

$$\forall \lambda > 0, \quad \forall u \in X \quad \lim_{h \rightarrow 0} \left(I + \frac{\lambda}{h}(I - U(h)) \right)^{-1} u = (I + \lambda A)^{-1} u.$$

ii) One can choose $u_0 \in X$, α satisfying (34) and a sequence $\{h_1^n, \dots, h_{N^n}^n\}$ such that

$$\sum_{i=1}^{N^n} h_i^n = T, \quad \lim_{n \rightarrow \infty} \max_{1 \leq i \leq N^n} h_i^n = 0$$

$$\lim_{n \rightarrow \infty} \|U(h_{N^n}^n) U(h_{N^n-1}^n) \dots U(h_1^n) u_0\|_\infty = +\infty.$$

Proof. The first statement of i) is immediate from the definition (35) and the fact that $S(h)$ is nonexpansive in X .

Let $u \in X$, and $u_h = [I + \lambda h^{-1}(I - U(h))]^{-1}u$, that is

$$(36) \quad u_h + \lambda h^{-1}(u_h - U(h)u_h) = u$$

or, by (35)

$$u_h + \lambda h^{-1}(u_h - S(h)u_h - \alpha(h)(u_0 - S(h)u_0)) = u,$$

which can be written as

$$(37) \quad u_h - \alpha(h)u_0 + \lambda h^{-1}(u_h - \alpha(h)u_0 - S(h)(u_h - \alpha(h)u_0)) = u - \alpha(h)u_0.$$

Setting $J_\lambda^h = [I + \lambda h^{-1}(I - S(h))]^{-1}$, (36) is equivalent to

$$(38) \quad u_h = \alpha(h)u_0 + J_\lambda^h(u - \alpha(h)u_0).$$

Since $S(h)$ is a linear semigroup of contractions, J_λ^h is also nonexpansive and satisfies (see for instance [9])

$$(39) \quad \lim_{h \rightarrow 0} J_\lambda^h u = (I + \lambda A)^{-1}u.$$

But now, since J_λ^h is nonexpansive, by (38)

$$|u_h - J_{\lambda}^h u_0| < 2\alpha(h) |u_0|.$$

This together with (39) and (34) proves the assertion i).

Now we choose a sequence of numbers $(h_i^n)_{1 \leq i \leq N^n}$ in $(0,1)$ satisfying

$$(40) \quad \sum_{i=1}^{N^n} h_i^n = T > 0$$

$$(41) \quad \forall n, i \neq j \Rightarrow h_i^n \neq h_j^n$$

$$(42) \quad \frac{T}{n+1} < \min_{1 \leq i \leq N^n} h_i^n < \max_{1 \leq i \leq N^n} h_i^n < \frac{T}{n}.$$

The conditions (41) - (42) insure that all the numbers

$\{h_i^n, 1 \leq i \leq N^n, n \geq 1\}$ are different. This will allow us to assign the values $\alpha(h_i^n)$ as we wish. We do it as follows.

Set $t_i^n = \sum_{k=0}^i h_k^n$, $t_0^n = 0$, and choose $u_0 \in C_0(\mathbb{R})$ such that

$$(43) \quad \theta_n = \sum_{i=1}^{N^n} |u_0(t_i^n) - u_0(t_{i-1}^n)|$$

tend to ∞ when n tends to ∞ . Actually any $u_0 \in C_0(\mathbb{R})$ whose total variation on $[0, T]$ is infinite will be convenient. Indeed θ_n is the total variation of the step function u_0^n defined by

$$\forall i = 1, \dots, N^n, \quad \forall t \in [t_{i-1}^n, t_i^n], \quad u_0^n(t) = u_0(t_i^n), \quad u_0^n(0) = u_0(0),$$

and u_0^n converges uniformly on $[0, T]$ to u_0 when n tends to ∞ .

Therefore the total variation of u_0^n cannot stay bounded if the variation of u_0 is infinite.

Now, for reasons that will become clear later we choose $\alpha: (0,1) \rightarrow \mathbb{R}$ as follows:

$$(44) \quad \begin{aligned} &\text{if } h = h_i^n, \quad \alpha(h) = \theta_n^{-1/2} \operatorname{sign} (u_0(t_i^n) - u_0(t_{i-1}^n)) \\ &\text{if } h \notin \{h_i^n; 1 \leq i \leq N^n, n \geq 1\}, \quad \alpha(h) = 0. \end{aligned}$$

Note that α satisfies (34).

We claim that with this choice, if we denote

$$f^n = U(h_{N^n}^n) U(h_{N^n-1}^n) \dots U(h_1^n) 0,$$

then $f^n(T)$ tends to ∞ when n tends to ∞ and therefore

$$(45) \quad \lim_{n \rightarrow \infty} \|f^n\|_{\infty} = \infty.$$

Indeed, if $f_k^n = U(h_k^n) U(h_{k-1}^n) \dots U(h_1^n) 0$ for $k \geq 1$ and $f_0^n \equiv 0$ by definition (35) we have

$$f_k^n = S(h_k^n) f_{k-1}^n + \alpha(h_k^n) (u_0 - S(h_k^n) u_0).$$

By induction, we check that

$$f_k^n = \sum_{i=1}^k \alpha(h_i^n) [S(t_k^n - t_i^n) u_0 - S(t_k^n - t_{i-1}^n) u_0]$$

so that

$$f^n = \sum_{i=1}^{N^n} \alpha(h_i^n) [S(T - t_i^n) u_0 - S(T - t_{i-1}^n) u_0].$$

But $S(T - t_i^n) u_0(x) = u_0(x - T + t_i^n)$, so that according to (44),

$$(46) \quad f^n(T) = \theta_n^{-1/2} \sum_{k=1}^{N^n} |u_0(t_k^n) - u_0(t_{k-1}^n)| = \theta_n^{1/2}.$$

From (43), (46) we obtain:

$$\lim_{n \rightarrow \infty} f^n(T) = \lim_{n \rightarrow \infty} \theta_n^{1/2} = \infty.$$

Remarks.

- i) The same negative conclusion could be reached by choosing

$$U(h)u = (I + hA)^{-1}u + \alpha(h)(u_0 - (I + hA)^{-1}u_0)$$

which is nothing but the resolvents of the operator $u \mapsto A(u - \alpha(h)u_0)$. Note that $U(h)$ as defined by (35) is the value at h of the semigroup generated by the operator $u \mapsto A(u - \alpha(h)u_0)$.

ii) According to this counterexample, the conclusion (23) dramatically fails as soon one switches from a regular step-size to a very slightly irregular path. In view of (41), (42) we need all the h_i^n to be different but they can be as close to each other as we pleased (see (41), (42)).

iii) It would be interesting to know whether there exists a condition stronger than (4) but weaker than (2) such that the expected conclusion (8) holds. Let us mention some positive results with variable step-size obtained in [10] under stability assumptions on the mappings $U(h)$.

iv) If B , C and $\overline{B+C}$ are infinitesimal generators of linear semigroups of contractions and if one sets

$$A = B + C$$

$$U(h) = (I+hB)^{-1} (I+hC)^{-1} \quad \text{or} \quad U(h) = S^B(h) S^C(h),$$

one easily verifies that assumption (11) holds. Since $\overline{S^{B+C}}(t) u_0$ is a mild solution of $\frac{du}{dt} + Au = 0$, $u(0) = u_0$ for all u_0 (see [1]), for both choices one has, as a consequence of Theorem 2.1

$$\lim_{n \rightarrow \infty} U(h_{N^n}^n) U(h_{N^{n-1}}^n) \dots U(h_1^n) u_0 = \overline{S^{B+C}}(t) u_0.$$

v) The same conclusion holds if $A = B + C$ where B is the infinitesimal generator of a linear semigroup of contractions and C is a continuous accretive operator defined on the whole space.

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20. Abstract (cont.)

where $\{U(h)\}_{h>0}$ is any family of contractions on X whose right-derivative at $h = 0$ coincides with the infinitesimal generator A of $S(t)$, that is

$$(2) \quad \forall x \in D(A) \quad \lim_{h \downarrow 0} \frac{x - U(h)x}{h} = Ax \quad (= \lim_{t \downarrow 0} \frac{x - S(t)x}{t}) .$$

Actually formula (1) extends to the case when $S(t)$ is a semigroup of nonlinear contractions "generated" by an m -accretive operator A and condition (2) may also be weakened to

$$(3) \quad \frac{I - U(h)}{h} \xrightarrow{h \downarrow 0} A \quad \text{in the sense of graphs.}$$

We look here at the same formula (1) when the regular step-size t/n is replaced by a variable step-size, namely

$$(4) \quad \lim_{n \rightarrow \infty} U(h_{N^n}^n) U(h_{N^{n-1}}^n) \dots U(h_1^n)x = S(t)x \quad \forall x \in \overline{D(A)}$$

where $\sum_{1 \leq i \leq N^n} h_i^n = t$ and $\lim_{n \rightarrow \infty} (\max_{1 \leq i \leq N^n} h_i^n) = 0$.

Surprisingly, it turns out that (4) fails under assumption (3); we exhibit a counterexample. However, we prove that (4) holds true under the stronger assumption (2).

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